İzmir Institute of Technology
MSE 222 Applied Mathematics for Materials Science and Engineering, Spring 2026
Midterm II - Solution Key

Name: ____________________

Student ID: ____________________

Duration: 105 Minutes

Grade Table

Question:	1	2	3	4	Total
Points:	6	8	5	5	24
Score:					

1. (a) (3 points) (WebWork) Let W be the subspace spanned by the vectors

$$u_1 = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \qquad \text{and} \qquad u_2 = \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix}.$$

Given

$$v = \begin{bmatrix} 4 \\ -13 \\ 7 \end{bmatrix} \in W,$$

find the coordinates of v with respect to the ordered basis $\{u_1, u_2\}$.

First, we observe that

$$\langle u_1, u_2 \rangle = 2 - 5 + 3 = 0,$$

so the vectors u_1 and u_2 are orthogonal. Therefore, if

$$v = c_1 u_1 + c_2 u_2,$$

then the coordinates of v are given by

$$c_1 = \frac{\langle v, u_1 \rangle}{\langle u_1, u_1 \rangle}, \qquad c_2 = \frac{\langle v, u_2 \rangle}{\langle u_2, u_2 \rangle}.$$

Computing the inner products for c_1, we get,

$$\langle v, u_1 \rangle = 8 + 13 + 21 = 42, \quad \langle u_1, u_1 \rangle = 2^2 + (-1)^2 + 3^2 = 14.$$

Hence

$$c_1 = \frac{42}{14} = 3.$$

Similarly,

$$\langle v, u_2 \rangle = 4 - 65 + 7 = -54, \quad \langle u_2, u_2 \rangle = 1^2 + 5^2 + 1^2 = 27.$$

Thus

$$c_2 = \frac{-54}{27} = -2.$$

Therefore,

$$v = 3u_1 - 2u_2.$$

So the coordinates of v with respect to the ordered basis $\{u_1, u_2\}$ are

$$\boxed{[v]_{\{u_1,u_2\}} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}}.$$

(b) (3 points) (WebWork) Let $\mathbb{P}_2$ be the vector space of all polynomials of degree at most 2. For $f, g \in \mathbb{P}_2$, define the inner product by

$$\langle f, g \rangle = f(-1)g(-1) + f(0)g(0) + f(1)g(1).$$

Let

$$u(x) = x^2 - 1 \qquad \text{and} \qquad v(x) = 2x^2 + 3x.$$

With respect to the given inner product, answer the following: Is the set $\{u, v\}$ orthogonal? Is it orthonormal?

We first compute the values of u and v at the points $-1, 0, 1$.

For

$$u(x) = x^2 - 1,$$

we have

$$u(-1) = 0, \qquad u(0) = -1, \qquad u(1) = 0.$$

For

$$v(x) = 2x^2 + 3x,$$

we have

$$v(-1) = 2 - 3 = -1, \qquad v(0) = 0, \qquad v(1) = 2 + 3 = 5.$$

Now compute the inner product:

$$\begin{aligned} \langle u, v \rangle &= u(-1)v(-1) + u(0)v(0) + u(1)v(1) \\ &= 0(-1) + (-1)(0) + 0(5) = 0. \end{aligned}$$

Therefore, u and v are orthogonal with respect to the given inner product, so the set $\{u, v\}$ is an orthogonal set. To check whether the set is orthonormal, we compute the norms. First,

$$\langle u, u \rangle = u^2(-1) + u^2(0) + u^2(1) = 0^2 + (-1)^2 + 0^2 = 1.$$

Hence,

$$\|u\| = \sqrt{\langle u, u \rangle} = 1.$$

Next,

$$\langle v, v \rangle = v^2(-1) + v^2(0) + v^2(1) = (-1)^2 + 0^2 + 5^2 = 26.$$

Hence,

$$\|v\| = \sqrt{\langle v, v \rangle} = \sqrt{26} \neq 1.$$

Although u and v are orthogonal, v does not have norm 1. Therefore, the set is not orthonormal.

$$\boxed{\{u, v\} \text{ is orthogonal but not orthonormal.}}$$

2. Answer the following questions. Explain each answer in two or three sentences.

(a) (2 points) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation that collapses all of space onto a plane through the origin. Explain why the matrix representation A of T must be singular.

Since T collapses $\mathbb{R}^3$ onto a plane, the image of T is at most two-dimensional. Hence the columns of A are linearly dependent, so A is singular.

(b) (2 points) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}.$$

This map sends every vector to the horizontal plane $z = 1$. Is T linear?

No. The map T is not linear because homogeneity does not hold.

(c) (2 points) The "*cyclic*" transformation T on $\mathbb{R}^3$ is defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_3 \\ x_1 \\ x_2 \end{bmatrix}$$

Let A be the matrix representation for T. Without computing the determinant, explain in words why $\det(A) \neq 0$.

The transformation T sends the standard basis vectors to the same basis vectors in a different order:

$$T(e_1) = e_2, \qquad T(e_2) = e_3, \qquad T(e_3) = e_1.$$

Thus, the columns of A are still linearly independent. Therefore, A is invertible, and hence $\det(A) \neq 0$.

(d) (2 points) Let $\{u_1, u_2\}$ be an orthonormal basis of $\mathbb{R}^2$ and $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a rotation. Is it true that $\{Tu_1, Tu_2\}$ is also an orthonormal basis of $\mathbb{R}^2$?

Yes. A rotation preserves lengths and angles. Since $\{u_1, u_2\}$ is an orthonormal set, $\{Tu_1, Tu_2\}$ is also an orthonormal set, therefore a basis of $\mathbb{R}^2$.

3. (5 points) Let V be the vector space spanned by the set of functions $\{1, 1+\cos x, 1+\sin x\}$ on the interval $(0, 2\pi)$. Define the inner product on V by

$$\langle f, g\rangle = \int_0^{2\pi} f(x)g(x)\,dx.$$

Apply the Gram–Schmidt process to the ordered set

$$\{1, 1+\cos x, 1+\sin x\}$$

and obtain an orthogonal basis for V.

Let

$$u_1(x) = 1, \qquad u_2(x) = 1+\cos x, \qquad u_3(x) = 1+\sin x.$$

We apply the Gram–Schmidt process to the ordered set $\{u_1, u_2, u_3\}$ to obtain an orthogonal set $\{w_1, w_2, w_3\}$. First, we set

$$w_1 = u_1 = 1.$$

Next,

$$w_2 = u_2 - \frac{\langle u_2, w_1\rangle}{\langle w_1, w_1\rangle} w_1$$

We compute

$$\langle u_2, w_1\rangle = \int_0^{2\pi} (1+\cos x)(1)\,dx = 2\pi, \quad \langle w_1, w_1\rangle = \int_0^{2\pi} 1\,dx = 2\pi.$$

Therefore,

$$w_2 = (1+\cos x) - \frac{2\pi}{2\pi}(1) = \cos x.$$

As the third and the last step,

$$w_3 = u_3 - \frac{\langle u_3, w_1\rangle}{\langle w_1, w_1\rangle} w_1 - \frac{\langle u_3, w_2\rangle}{\langle w_2, w_2\rangle} w_2.$$

We compute

$$\langle u_3, w_1\rangle = \int_0^{2\pi} (1+\sin x)(1)\,dx = 2\pi.$$

Also,

$$\langle u_3, w_2\rangle = \int_0^{2\pi} (1+\sin x)\cos x\,dx = \int_0^{2\pi} \cos x\,dx + \int_0^{2\pi} \sin x\cos x\,dx = 0$$

and

$$\langle w_2, w_2\rangle = \int_0^{2\pi} \cos^2 x\,dx = \pi.$$

Hence,

$$w_3 = (1+\sin x) - \frac{2\pi}{2\pi}(1) - \frac{0}{\pi}\cos x = \sin x.$$

Therefore, an orthogonal basis for V is

$$\boxed{\{1, \cos x, \sin x\}}.$$

4. (5 points) Find the Fourier series expansion of $f(x) = |x|$ on the interval $(-\pi, \pi)$.

The Fourier series of $f(x) = |x|$ on $(-\pi, \pi)$ is of the form

$$a_0 + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right),$$

where

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |x|\, dx, \qquad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} |x| \cos nx\, dx, \qquad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} |x| \sin nx\, dx.$$

Since $f(x) = |x|$ is even and $\sin nx$ is odd, the product $|x| \sin nx$ is odd. Hence,

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} |x| \sin nx\, dx = 0$$

for all $n \geq 1$. Also,

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |x|\, dx = \frac{1}{2\pi} \cdot 2 \int_0^{\pi} x\, dx = \frac{\pi}{2}.$$

Now, the product $|x| \cos nx$ is even so we can write

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} |x| \cos nx\, dx = \frac{2}{\pi} \int_0^{\pi} x \cos nx\, dx.$$

Using integration by parts,

$$\int_0^{\pi} x \cos nx\, dx = \left[\frac{x \sin nx}{n}\right]_0^{\pi} - \frac{1}{n} \int_0^{\pi} \sin nx\, dx.$$

Since $\sin(n\pi) = 0$ for all $n \geq 1$, we get

$$\int_0^{\pi} x \cos nx\, dx = -\frac{1}{n} \left[-\frac{\cos nx}{n}\right]_0^{\pi} = \frac{(-1)^n - 1}{n^2}.$$

Thus,

$$a_n = \frac{2}{\pi} \frac{(-1)^n - 1}{n^2}.$$

Therefore, the Fourier series expansion is

$$\boxed{\frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2\big((-1)^n - 1\big)}{\pi n^2} \cos nx}.$$