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İzmir Institute of Technology
MSE 222 Applied Mathematics for Materials Science and Engineering, Spring 2026
Midterm I - Solution Key

Name: ______________________________

Student ID: ______________________________

Duration: 105 Minutes

Grade Table

Question:	1	2	3	4	Total
Points:	4	11	4	5	24
Score:					

1. (a) (2 points) (WebWork) Find a basis of the subspace of $\mathbb{R}^4$ consisting of all vectors of the form

$$\begin{bmatrix} x_1 \\ 3x_1 + x_2 \\ 4x_1 - 2x_2 \\ -7x_1 - 7x_2 \end{bmatrix}, \quad x_1, x_2 \in \mathbb{R}.$$

A general vector in the given subspace can be written as

$$\begin{bmatrix} x_1 \\ 3x_1 + x_2 \\ 4x_1 - 2x_2 \\ -7x_1 - 7x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 3 \\ 4 \\ -7 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ -2 \\ -7 \end{bmatrix}.$$

Thus the subspace is spanned by the vectors

$$\begin{bmatrix} 1 \\ 3 \\ 4 \\ -7 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 1 \\ -2 \\ -7 \end{bmatrix}.$$

Since these two vectors are not scalar multiples of each other, they are linearly independent. Therefore, they form a basis. Hence, a basis for the subspace is

$$\boxed{\left\{ \begin{bmatrix} 1 \\ 3 \\ 4 \\ -7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \\ -7 \end{bmatrix} \right\}.}$$

(b) (2 points) (WebWork) Let

$$\mathbf{a}_1 = \begin{bmatrix} 7 \\ 8 \\ -9 \end{bmatrix}, \qquad \mathbf{a}_2 = \begin{bmatrix} -4 \\ 8 \\ 4 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 18 \\ 8 \\ -22 \end{bmatrix}.$$

Is $\mathbf{b}$ in the span of $\mathbf{a}_1$ and $\mathbf{a}_2$? Provide details.

To determine whether $\mathbf{b}$ is in the span of $\mathbf{a}_1$ and $\mathbf{a}_2$, we must check whether there exist scalars $c_1, c_2 \in \mathbb{R}$ such that

$$c_1\mathbf{a}_1 + c_2\mathbf{a}_2 = \mathbf{b}, \quad (\star).$$

That is, we solve

$$c_1 \begin{bmatrix} 7 \\ 8 \\ -9 \end{bmatrix} + c_2 \begin{bmatrix} -4 \\ 8 \\ 4 \end{bmatrix} = \begin{bmatrix} 18 \\ 8 \\ -22 \end{bmatrix}.$$

This system gives the augmented matrix

$$\left[\begin{array}{cc|c} 7 & -4 & 18 \\ 8 & 8 & 8 \\ -9 & 4 & -22 \end{array}\right].$$

Performing several elementary row operations, the above augmented form can be reduced to the following row-echelon form

$$\left[\begin{array}{cc|c} 7 & -4 & 18 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right].$$

Since there is no row of the form

$$\left[\begin{array}{cc|c} 0 & 0 & c \end{array}\right] \qquad \text{with } c \neq 0,$$

the system is consistent. Therefore, there exists c_1 and c_2 such that $(\star)$ holds true, hence $\boxed{\mathbf{b} \in \operatorname{span}\{\mathbf{a}_1, \mathbf{a}_2\}}$.

2. Answer the following questions briefly. Explain each answer in one or two sentences.

(a) (3 points) Let A be a 5×8 matrix with $\operatorname{rank}(A) = 5$ and let $b \in \mathbb{R}^5$. Explain why the system must be consistent. Explain why the system $Ax = b$ must have infinitely many solutions.

Since $\operatorname{rank}(A) = 5$, the matrix A has a pivot in every row. Therefore, for every $b \in \mathbb{R}^5$, the augmented system $Ax = b$ cannot produce a contradictory row, so the system is always consistent.

Also, A has 8 columns and rank 5, so there are $8 - 5 = 3$ free variables. Hence the system has infinitely many solutions.

(b) (2 points) Suppose that a square matrix A has a nonzero vector in its null space. Can you conclude that A is row-equivalent to the identity matrix?

No. If A has a nonzero vector in its null space, then the homogeneous system $Ax = 0$ has a nontrivial solution. This means that the columns of A are linearly dependent, so A cannot have a pivot in every column.

Therefore, A cannot be row-equivalent to the identity matrix.

(c) (2 points) Suppose that the rows of a square matrix A are linearly dependent. Is A invertible?

No. If the rows of a square matrix A are linearly dependent, then the row-echelon form of A must contain a zero row. Hence, there cannot be a pivot in every row.

Therefore, A cannot be row-equivalent to the identity matrix. Since a square matrix is invertible if and only if it is row-equivalent to I, it follows that A is not invertible.

(d) (2 points) Let E be a product of a sequence of elementary matrices. Is it true that the homogeneous system $Ex = 0$ has a unique solution?

Yes. Every elementary matrix is invertible, and the product of invertible matrices is again invertible. Therefore, E is invertible.

Hence the homogeneous system $Ex = 0$ has a unique solution which is the trivial solution.

(e) (2 points) Let $\{v_1, v_2, v_3\}$ be a basis for a vector space V, and let $v_4 \in V$. If v_4 is added to this set, will the new set still be a basis for V? Will it be a spanning set for V?

The new set will still be a spanning set for V, because $\{v_1, v_2, v_3\}$ already spans V, and adding one more vector cannot destroy that property.

However, the new set will not be a basis for V, because $v_4 \in V$ can already be written as a linear combination of v_1, v_2, v_3. Thus the enlarged set is linearly dependent.

3. (a) (2 points) Determine whether the following matrix is row-equivalent to the 3×3 identity matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 6 & 10 \end{bmatrix}.$$

To determine whether A is row-equivalent to the 3×3 identity matrix, we row reduce A and check whether its row-echelon form is I_3. We begin applying the row operations

$$R_2 \leftarrow R_2 - 2R_1, \qquad R_3 \leftarrow R_3 - 3R_1.$$

Then

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 6 & 10 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Next, apply

$$R_3 \leftarrow R_3 - R_2.$$

This gives

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus, a row-echelon form of A is

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since this matrix has a zero row, A has fewer than 3 pivots. Therefore,

$$\boxed{A \text{ is not row-equivalent to the } 3 \times 3 \text{ identity matrix.}}$$

(b) (2 points) Express the following matrix as a product of elementary matrices:

$$A = \begin{bmatrix} 3 & 2 & -3 \\ 1 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

To express A as a product of elementary matrices, we first row reduce A to the identity matrix:

$$A = \begin{bmatrix} 3 & 2 & -3 \\ 1 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 0 & 0 \\ 3 & 2 & -3 \\ -1 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 - 3R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -3 \\ -1 & 0 & 1 \end{bmatrix} \xrightarrow{R_3 \leftarrow R_3 + R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_2 \leftarrow \frac{1}{2} R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{3}{2} \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 + \frac{3}{2} R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I.$$

Thus,

$$E_5E_4E_3E_2E_1A = I,$$

and hence

$$A = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}E_5^{-1}.$$

The corresponding inverse elementary matrices are

$$E_1^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix},$$

$$E_4^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad E_5^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{3}{2} \\ 0 & 0 & 1 \end{bmatrix}.$$

Therefore,

$$\boxed{A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{3}{2} \\ 0 & 0 & 1 \end{bmatrix}.}$$

4. (5 points) Consider the following system of linear equations:

$$\begin{cases} x_1 + x_2 + x_3 = 2, \\ 2x_1 + 3x_2 + x_3 = 3, \\ x_1 + 2x_2 = 1, \\ 3x_1 + 5x_2 + x_3 = 4. \end{cases}$$

Using Gauss–Jordan elimination, determine whether the system is consistent or inconsistent. If it is consistent, determine whether it has a unique solution or infinitely many solutions. Also determine the dimension of the solution set.

We write the augmented matrix of the system as

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & 3 & 1 & 3 \\ 1 & 2 & 0 & 1 \\ 3 & 5 & 1 & 4 \end{array}\right].$$

Now we perform Gauss–Jordan elimination:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & 3 & 1 & 3 \\ 1 & 2 & 0 & 1 \\ 3 & 5 & 1 & 4 \end{array}\right] \xrightarrow[R_4 \leftarrow R_4 - 3R_1]{\substack{R_2 \leftarrow R_2 - 2R_1 \\ R_3 \leftarrow R_3 - R_1}} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 1 & -1 & -1 \\ 0 & 2 & -2 & -2 \end{array}\right]$$

$$\xrightarrow[R_4 \leftarrow R_4 - 2R_2]{R_3 \leftarrow R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 \leftarrow R_1 - R_2} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Thus, the reduced row-echelon form is

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Since there is no row of the form

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & c \end{array}\right] \qquad \text{with } c \neq 0,$$

the system is consistent.

From the reduced row-echelon form, the pivot variables are x_1 and x_2, while x_3 is a free variable. Therefore, the system has infinitely many solutions.

Let

$$x_3 = t, \qquad t \in \mathbb{R}.$$

Then the first two rows give

$$x_1 + 2x_3 = 3 \quad \Longrightarrow \quad x_1 = 3 - 2t,$$
$$x_2 - x_3 = -1 \quad \Longrightarrow \quad x_2 = t - 1.$$

Hence the solution set is

$$\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 - 2t \\ t - 1 \\ t \end{bmatrix} : t \in \mathbb{R} \right\}.$$

Since there is one free variable, the dimension of the solution set is 1. Therefore,

$$\boxed{\text{the system is consistent, it has infinitely many solutions, and the solution set has dimension 1.}}$$