22.06.2026

İzmir Institute of Technology
MSE 222 Applied Mathematics for Materials Science and Engineering, Spring 2026
Final Exam - Solution Key

Name: ______________________________

Student ID: ______________________________

Duration: 120 Minutes

Grade Table

Question:	1	2	3	Total
Points:	10	18	20	48
Score:				

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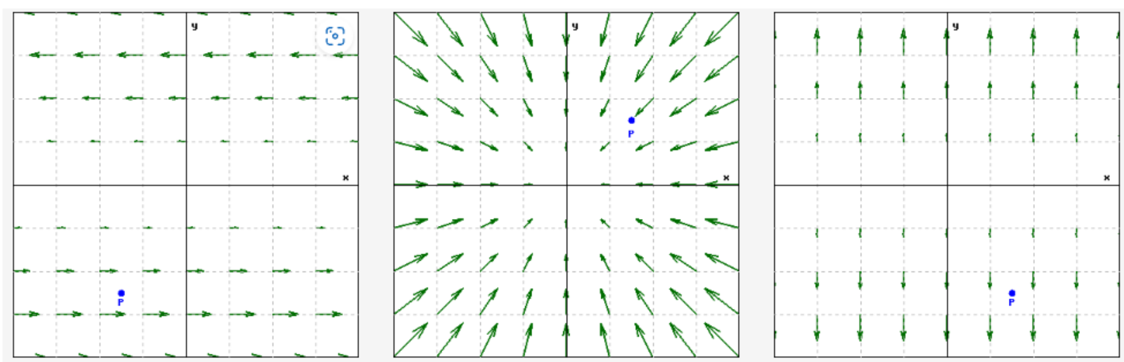
1. (WebWork)

   (a) (3 points) Determine whether the divergence of each vector field at the indicated point P is positive, negative or zero.

   Hint: Draw a small rectangular region around the point P and think about the total flux.

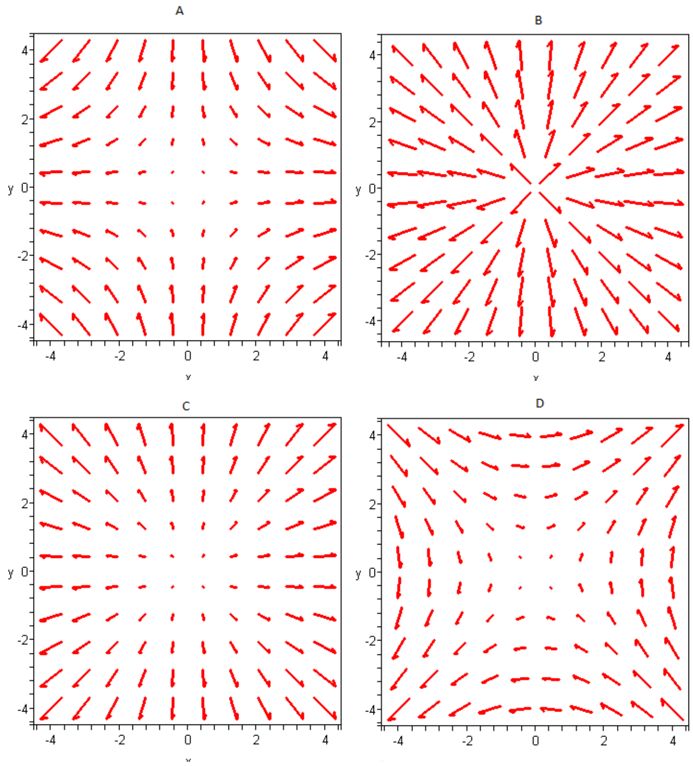


- First figure: divergence is zero.
  In the first figure, near the point P, the vector field does not appear to produce a net outward or inward flux through a small rectangle around P. The amount of flow entering the rectangle is equal to the amount of flow leaving it. Therefore, the total outward flux is zero, hence the divergence is zero.

- Second figure: divergence is negative.
  If we draw a small rectangle around P, some flow enters and some flow leaves the rectangle, but the incoming flow is larger than the outgoing flow. Therefore, the total outward flux is negative, hence the divergence is negative.

- Third figure: divergence is positive.
  By a similar reasoning, the outgoing flow is larger than the incoming flow. Therefore, the total outward flux is positive, so the divergence is positive.

(b) (4 points) Match the functions f with the plots of their gradient vector fields labeled as $A - D$.



1. $f(x, y) = x^2 - y^2$
2. $f(x, y) = xy$
3. $f(x, y) = x^2 + y^2$
4. $f(x, y) = \sqrt{x^2 + y^2}$

1. For $f(x, y) = x^2 - y^2$, we have
$$\nabla f = (2x, -2y).$$
The vectors point away from the y-axis horizontally, but toward the x-axis vertically. This matches figure A.
2. For $f(x, y) = xy$, we have
$$\nabla f = (y, x).$$
The direction of the vector depends on both coordinates in a crossed way. This gives the saddle-like pattern in figure D.
3. For $f(x, y) = x^2 + y^2$, we have
$$\nabla f = (2x, 2y).$$
The vectors point directly away from the origin, and they get longer as we move farther from the origin. This matches figure C.
4. For $f(x, y) = \sqrt{x^2 + y^2}$, we have
$$\nabla f = \left(\frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}} \right).$$
The vectors point away from the origin and have the same length except at the origin, where the gradient is not defined. This matches figure B.

(c) (3 points) Let $u = u(x, y, t)$ denote the concentration of a chemical substance in a moving fluid. Suppose that the substance is transported by the motion of the fluid and also diffuses in the medium, so that the total flux is formulated by

$$\mathbf{q} = u\mathbf{v} - k\nabla u,$$

where $\mathbf{v}$ is the velocity field of the fluid and $k > 0$ is the diffusion coefficient.

Starting from the continuity equation

$$u_t = -\nabla \cdot \mathbf{q} + f,$$

derive the corresponding partial differential equation, if the fluid velocity is constant and given by $\mathbf{v} = (1, 0)$ and the diffusion coefficient is $k = 1$.

We are given

$$\mathbf{q} = u\mathbf{v} - k\nabla u.$$

So

$$\nabla \cdot \mathbf{q} = \nabla \cdot (u\mathbf{v}) - \nabla \cdot (k\nabla u).$$

Since $\mathbf{v} = (1, 0)$, we have $u\mathbf{v} = (u, 0)$. Therefore,

$$\nabla \cdot (u\mathbf{v}) = u_x.$$

Also, since $k = 1$,

$$\nabla \cdot (k\nabla u) = \nabla \cdot \nabla u = \Delta u = u_{xx} + u_{yy}.$$

Consequently,

$$\nabla \cdot \mathbf{q} = u_x - (u_{xx} + u_{yy}).$$

Substituting into the continuity equation gives

$$u_t + u_x - u_{xx} - u_{yy} = f.$$

2. Answer the following questions. If necessary, explain your answer in two or three sentences.

(a) (3 points) A river flows steadily from west to east. The speed of the water is constant everywhere, and there are no sources or sinks in the river.

Would you expect the flow to be incompressible? Explain.

Since the velocity field is constant, its divergence is zero. Therefore, the flow is incompressible.

(b) (3 points) A small amount of gas is released in a room and its concentration is represented by a function $c(x, y, t)$. At a certain point, the gradient ∇c points toward the north-east.

In which direction would the gas tend to diffuse from that point? Explain.

The gradient ∇c points toward higher concentration. Since diffusion occurs from higher concentration to lower concentration, the gas moves in the direction opposite to ∇c. Therefore, the gas would tend to diffuse toward the south-west.

(c) (3 points) A thin metal plate is heated at a point. Near this heat source, the temperature increases toward the source and decreases away from it.

Would you expect the Laplacian of the temperature function to be positive, negative, or zero in a small region around the heat source? Explain.

The Laplacian is the divergence of the gradient:

$$\Delta T = \nabla \cdot (\nabla T).$$

Near the heat source, the temperature increases toward the source, so the gradient field ∇T points toward the source. Thus, in a small region around the source, the gradient vectors are directed inward. This corresponds to negative divergence of the gradient field. Therefore, we expect $\Delta T < 0$.

(d) (3 points) A small amount of dye is released in a thin layer of water. After some time, the dye is observed to move mainly to the right, but at the same time it becomes less concentrated and spreads over a larger region in the water.

Would a pure transport model be enough to describe this behavior? Explain your answer in words. Also suggest a flux field for the dye concentration $c(x, y, t)$ that describes this behavior.

The dye moves mainly to the right, so there is transport. But the dye also spreads out and becomes less concentrated, so there is diffusion as well. Therefore, the answer is no, a pure transport model would not be enough.

A suitable flux field is

$$\mathbf{q} = c\mathbf{v} - k\nabla c, \quad \text{where } \mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \text{ and } k > 0 \text{ is a constant.}$$

The term $c\mathbf{v}$ moves the dye to the right. The term $-k\nabla c$ makes the dye spread from regions of high concentration to regions of low concentration.

(e) (3 points) Let $D = [0, 2] \times [0, 1]$ be the rectangular region in $\mathbb{R}^2$. Consider the vector field given by

$$\mathbf{F}(x, y) = x^2 y\mathbf{i} + \sin y\mathbf{j}.$$

Write a double integral that expresses the outward flux of the vector field across the boundary of D.

Hint: Use Green's theorem.

By Green's theorem, the outward flux integral can be written as

$$\iint_D \nabla \cdot \mathbf{F}\, dA.$$

Now compute the divergence:

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(x^2 y) + \frac{\partial}{\partial y}(\sin y) = 2xy + \cos y.$$

Since $D = [0, 2] \times [0, 1]$, the outward flux is given by

$$\int_0^2 \int_0^1 (2xy + \cos y)\, dy\, dx.$$

(f) (3 points) Suppose heat is diffusing along a thin wire of length one unit, represented by the interval $(0, 1)$. Let $u(x, t)$ denote the temperature of the wire at position x and time t. The left end of the wire is kept at zero temperature, while the right end is insulated, so that there is no flux across the right endpoint.

If you wanted to write a partial differential equation that models this heat diffusion process, what boundary conditions would you choose?

The left end is kept at zero temperature, so we impose $u(0, t) = 0$. The right end is insulated, meaning that there is no heat flux there. Since heat flux is proportional to $-u_x$, no flux at $x = 1$ gives $u_x(1, t) = 0$.

3. (20 points) Solve the following initial-boundary value problem for the heat equation:

$$\begin{cases} u_t = 100u_{xx}, & 0 < x < 1, \quad t > 0, \\ u(0,t) = 0, \quad u(1,t) = 0, & t > 0, \\ u(x,0) = \sin(2\pi x) - \sin(5\pi x), & 0 \le x \le 1. \end{cases}$$

Step 1. Separation of variables. Assume

$$u(x,t) = X(x)T(t).$$

Substituting into $u_t = 100u_{xx}$, we get

$$X(x)T'(t) = 100X''(x)T(t).$$

Dividing by $100X(x)T(t)$, we obtain

$$\frac{T'(t)}{100T(t)} = \frac{X''(x)}{X(x)} = \lambda.$$

Thus, we get the two ODEs

$$X'' - \lambda X = 0, \qquad T' - 100\lambda T = 0.$$

Step 2. Solve the spatial ODE. The boundary conditions give

$$X(0) = 0, \qquad X(1) = 0.$$

So we solve the following boundary value problem

$$\begin{cases} X'' - \lambda X = 0, \quad x \in (0,1), \\ X(0) = X(1) = 0. \end{cases}$$

The characteristic equation for $X'' - \lambda X = 0$ is $r^2 - \lambda = 0$.

- **Case 1: Real distinct roots.** If $\lambda > 0$, then $r_{1,2} = \pm\sqrt{\lambda}$. Thus,

  $$X(x) = Ae^{\sqrt{\lambda}x} + Be^{-\sqrt{\lambda}x}.$$

  Using $X(0) = 0$ and $X(1) = 0$, we get $A = B = 0$, so only the trivial solution.
- **Case 2: Repeated root.** If $\lambda = 0$, then $r = 0$. Thus,

  $$X(x) = A + Bx.$$

  Using $X(0) = 0$ and $X(1) = 0$, we again get only the trivial solution.
- **Case 3: Pure imaginary roots.** If $\lambda < 0$, then $r_{1,2} = i\sqrt{-\lambda}$. Thus,

  $$X(x) = A\cos(\sqrt{-\lambda}x) + B\sin(\sqrt{-\lambda}x).$$

  Imposing the boundary condition $X(0) = 0$ gives $A = 0$, so the general solution becomes $X(x) = B\sin(\sqrt{-\lambda}x)$. The other condition $X(1) = 0$ gives

  $$B\sin(\sqrt{-\lambda}) = 0.$$

  For a nontrivial solution, we need

  $$\sin(\sqrt{-\lambda}) = 0.$$

  Hence,

  $$\sqrt{-\lambda} = n\pi, \qquad n = 1, 2, 3, \dots$$

  which implies

  $$\lambda_n = -n^2\pi^2, \qquad X_n(x) = \sin(n\pi x).$$

Step 3. Solve the time ODE.

The time ODE is

$$T' - 100\lambda T = 0.$$

Since $\lambda_n = -n^2\pi^2$, we obtain

$$T_n' - 100(-n^2\pi^2)T_n = 0.$$

Thus,

$$T_n' + 100n^2\pi^2 T_n = 0.$$

Therefore,

$$T_n(t) = c_n e^{-100n^2\pi^2 t}.$$

Step 4. Use superposition to write the general solution.
For each $n = 1, 2, 3, \ldots$, we have

$$X_n(x) = \sin(n\pi x), \qquad T_n(t) = c_n e^{-100n^2\pi^2 t}.$$

Therefore, by superposition, the general solution is

$$u(x,t) = \sum_{n=1}^{\infty} X_n(x) T_n(t) = \sum_{n=1}^{\infty} c_n e^{-100n^2\pi^2 t} \sin(n\pi x).$$

Step 5. Apply the initial condition. Setting $t = 0$ on the general solution and using the initial condition, we have

$$\sum_{n=1}^{\infty} c_n \sin(n\pi x) = \sin(2\pi x) - \sin(5\pi x).$$

Hence, comparing the Fourier sine coefficients, we obtain

$$c_2 = 1, \qquad c_5 = -1,$$

and

$$c_n = 0 \qquad \text{for } n \neq 2, 5.$$

Therefore, the solution to the given initial-boundary value problem is

$$\boxed{u(x,t) = e^{-400\pi^2 t} \sin(2\pi x) - e^{-2500\pi^2 t} \sin(5\pi x).}$$