12.05.2026

İzmir Institute of Technology
Math 265 Basic Linear Algebra, Spring 2026
Midterm II - Solution Key

Name: ____________________________________

Student ID: ____________________________________

Duration: 120 Minutes

Grade Table

Question:	1	2	3	4	Total
Points:	8	12	9	7	36
Score:					

1. (a) (4 points) (WebWork) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that first reflects points across the y-axis, then stretches them vertically by a factor of 2 and then rotates them counterclockwise by $\frac{\pi}{2}$ radians. Find the standard matrix A for T.

The transformation is a composition of three transformations:

$$T = \text{rotation} \circ \text{vertical stretch} \circ \text{reflection}.$$

First, reflection across the y-axis has matrix

$$A_1 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then, a vertical stretch by a factor of 2 has matrix

$$A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

Finally, a counterclockwise rotation by $\frac{\pi}{2}$ has matrix

$$A_3 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Since the transformations are applied in the order reflection, then stretch, then rotation, the standard matrix is

$$A = A_3 A_2 A_1.$$

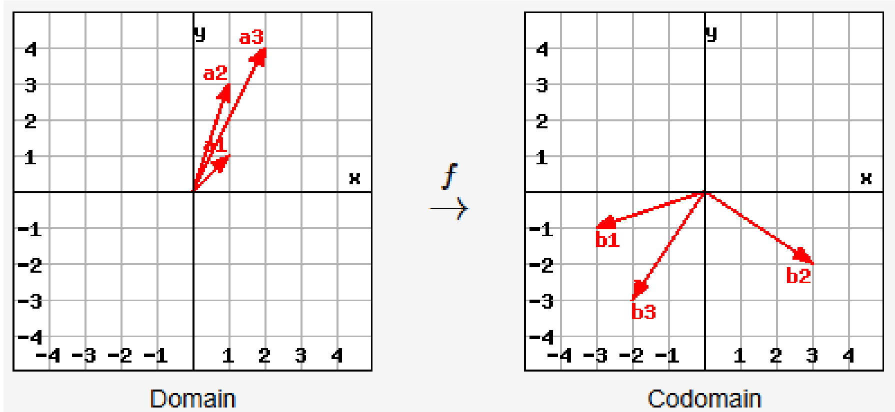
Thus,

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ -1 & 0 \end{bmatrix}.$$

Hence, the standard matrix for T is

$$\boxed{A = \begin{bmatrix} 0 & -2 \\ -1 & 0 \end{bmatrix}}.$$

(b) (4 points) Suppose $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a function and $f(\mathbf{a}_n) = \mathbf{b}_n$ for $n = 1, 2, 3$.



Is it true that f is additive? Explain.

No. From the diagram, we can see that

$$\mathbf{a}_1 + \mathbf{a}_2 = \mathbf{a}_3.$$

If f were additive, then we would have

$$f(\mathbf{a}_1 + \mathbf{a}_2) = f(\mathbf{a}_1) + f(\mathbf{a}_2), \quad (\star).$$

Since $\mathbf{a}_1 + \mathbf{a}_2 = \mathbf{a}_3$, from left-hand side of $(\star)$ we get

$$f(\mathbf{a}_1 + \mathbf{a}_2) = f(\mathbf{a}_3) = \mathbf{b}_3 = \begin{bmatrix} -2 \\ -3 \end{bmatrix}.$$

However, from the right-hand side of $(\star)$,

$$f(\mathbf{a}_1) + f(\mathbf{a}_2) = \mathbf{b}_1 + \mathbf{b}_2 = \begin{bmatrix} 0 \\ -3 \end{bmatrix}.$$

Therefore, $(\star)$ is not true and f is not additive.

2. Answer the following questions in two or three sentences. Explain your reasoning clearly.

   (a) (2 points) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation that collapses all of space onto a plane through the origin. Explain why the matrix representation A of T must be singular.

   Since T collapses $\mathbb{R}^3$ onto a plane, the image of T is at most two-dimensional. Hence the columns of A are linearly dependent, so A is singular.

   (b) (2 points) The cyclic transformation T on $\mathbb{R}^4$ is defined by

   $$T(x_1, x_2, x_3, x_4) = (x_4, x_1, x_2, x_3).$$

   Let A be the matrix representation of T. Without computing the determinant, explain why $\det(A) \neq 0$.

   The transformation T sends the standard basis vectors to the same basis vectors in a different order:

   $$T(e_1) = e_2, \qquad T(e_2) = e_3, \qquad T(e_3) = e_4, \quad T(e_4) = e_1.$$

   Thus, the columns of A are still linearly independent. Therefore, A is invertible, and hence $\det(A) \neq 0$.

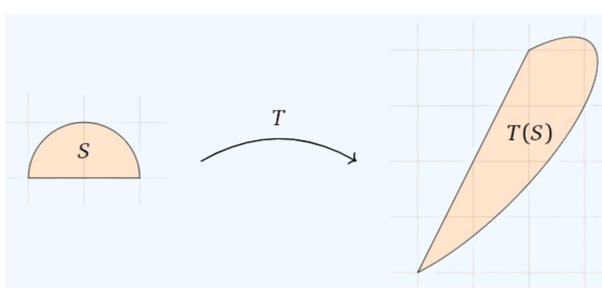
   (c) (2 points) Is it true that every square matrix is similar to itself?

   Yes. Every square matrix A can be expressed in the form

   $$A = I^{-1} A I,$$

   where I is the identity matrix which is nonsingular. Hence A is similar to itself.

   (d) (2 points) Let S be the semicircular region of radius 1 and $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation with matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$. What is the area of $T(S)$?



   $|\det(A)| = |1 - 4| = 3$. Therefore, the area of the region S is scaled by a factor of 3. Hence,

   $$\text{Area of } T(S) = 3 \times \text{Area of S} = \frac{3\pi}{2}.$$

(e) (2 points) Let V be the space of all infinitely differentiable functions, and let $D : V \to V$ be the differentiation operator. Define $L = D - I$, where I is the identity mapping. Is L one-to-one?

No.

$$L(f) = 0 \Rightarrow (D - I)f = 0 \Rightarrow f' = f.$$

This equation has a nonzero solution, for example $f(x) = e^x$. Hence $\mathcal{N}(L) \neq \{0_V\}$, so L is not one-to-one.

(f) (2 points) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a mapping that sends every vector into the horizontal plane $z = 1$. Is T linear?

No. $T(\mathbf{0})$ lies on the plane $z = 1$, so $T(\mathbf{0}) \neq \mathbf{0}$. Therefore, T is not linear.

3. Let $\mathbb{P}_3$ denote the vector space of all polynomials with real coefficients of degree at most 3, and let

$$\mathcal{S} = \{1, x, x^2, x^3\}$$

be its ordered basis. We represent

$$p(x) = ax^3 + bx^2 + cx + d \in \mathbb{P}_3$$

by the coordinate vector

$$[p]_{\mathcal{S}} = \begin{bmatrix} d \\ c \\ b \\ a \end{bmatrix} \in \mathbb{R}^4.$$

For example, if $p(x) = -3x^3 + x - 2 \in \mathbb{P}_3$, then

$$[p]_{\mathcal{S}} = \begin{bmatrix} -2 \\ 1 \\ 0 \\ -3 \end{bmatrix}.$$

Similarly, polynomials in $\mathbb{P}_2$ are represented as vectors in $\mathbb{R}^3$ with respect to the ordered basis $\{1, x, x^2\}$. Using the vector representations of polynomials described above, answer the following questions.

(a) (2 points) Write each basis element in $\mathcal{S}$ in vector form.

The ordered basis is

$$\mathcal{S} = \{1, x, x^2, x^3\}.$$

Using the given coordinate convention, we have

$$[1]_{\mathcal{S}} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad [x]_{\mathcal{S}} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad [x^2]_{\mathcal{S}} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad [x^3]_{\mathcal{S}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

(b) (4 points) Consider the differentiation operator

$$\frac{d}{dx} : \mathbb{P}_3 \to \mathbb{P}_2, \qquad p \mapsto p'.$$

Let D be the matrix representation of this operator with respect to the vector representations described above. Find D.

Let

$$p(x) = ax^3 + bx^2 + cx + d.$$

Then

$$[p]_{\mathcal{S}} = \begin{bmatrix} d \\ c \\ b \\ a \end{bmatrix}.$$

The derivative is

$$p'(x) = 3ax^2 + 2bx + c.$$

With respect to the ordered basis $\{1, x, x^2\}$ of $\mathbb{P}_2$, this is represented by

$$[p'] = \begin{bmatrix} c \\ 2b \\ 3a \end{bmatrix}.$$

Thus, we want a matrix D such that

$$D \begin{bmatrix} d \\ c \\ b \\ a \end{bmatrix} = \begin{bmatrix} c \\ 2b \\ 3a \end{bmatrix}.$$

To construct D, we look at the images of the standard basis vectors of $\mathbb{R}^4$.

$$De_1 = D\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad De_2 = D\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad De_3 = D\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \quad De_4 = D\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}.$$

Therefore, the columns of D are De_1, De_2, De_3, De_4. Hence

$$D = \begin{bmatrix} | & | & | & | \\ De_1 & De_2 & De_3 & De_4 \\ | & | & | & | \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

Thus,

$$\boxed{D = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}}.$$

(c) (3 points) Find the null space of D.
Differentiation sends constant polynomials to zero. A constant polynomial has the form

$$p(x) = d.$$

With respect to the basis $\mathcal{S} = \{1, x, x^2, x^3\}$, its coordinate vector is

$$[p]_{\mathcal{S}} = \begin{bmatrix} d \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Hence,

$$\mathcal{N}(D) = \left\{ \begin{bmatrix} d \\ 0 \\ 0 \\ 0 \end{bmatrix} : d \in \mathbb{R} \right\}.$$

Equivalently,

$$\boxed{\mathcal{N}(D) = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}.}$$

4. Let $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2\}$ be the standard basis for $\mathbb{R}^2$ and $\mathcal{F} = \{\mathbf{v}_1, \mathbf{v}_2\}$ be another ordered basis for $\mathbb{R}^2$, where

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation defined by

$$L(\mathbf{x}) = \begin{bmatrix} 2x_1 + x_2 \\ x_1 + 3x_2 \end{bmatrix}.$$

(a) (3 points) Find the matrix representation of L with respect to the standard basis $\mathcal{E}$.
With respect to the standard basis $\mathcal{E}$, the columns of the matrix are $L(\mathbf{e}_1)$ and $L(\mathbf{e}_2)$:

$$L(\mathbf{e}_1) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad L(\mathbf{e}_2) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

Therefore, the standard matrix A of L is

$$A = [L]_{\mathcal{E}} = \begin{bmatrix} | & | \\ L(\mathbf{e}_1) & L(\mathbf{e}_2) \\ | & | \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}.$$

(b) (4 points) Find the matrix representation of L with respect to the basis $\mathcal{F}$.
Let A be the matrix representation of L with respect to the standard basis $\mathcal{E}$, and B be the matrix representation of L with respect to the basis $\mathcal{F}$. Then A and B are related by the similarity relation

$$B = P^{-1}AP,$$

where P is the transition matrix from $\mathcal{F}$-coordinates to $\mathcal{E}$-coordinates. Since

$$\mathcal{F} = \{\mathbf{v}_1, \mathbf{v}_2\}, \qquad \mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix},$$

the transition matrix P is formed by placing the basis vectors of $\mathcal{F}$ as columns:

$$P = \begin{bmatrix} | & | \\ \mathbf{v}_1 & \mathbf{v}_2 \\ | & | \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}.$$

Using Gauss-Jordan, it can be obtained that

$$P^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}.$$

Therefore,

$$B = P^{-1}AP = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}.$$

Performing matrix multiplication, we find

$$\boxed{B = [L]_{\mathcal{F}} = \begin{bmatrix} \frac{10}{3} & \frac{1}{3} \\ \frac{5}{3} & \frac{5}{3} \end{bmatrix}}.$$