İzmir Institute of Technology
Math 265 Basic Linear Algebra, Spring 2026
Midterm I - Solution Key

Name: ______________________

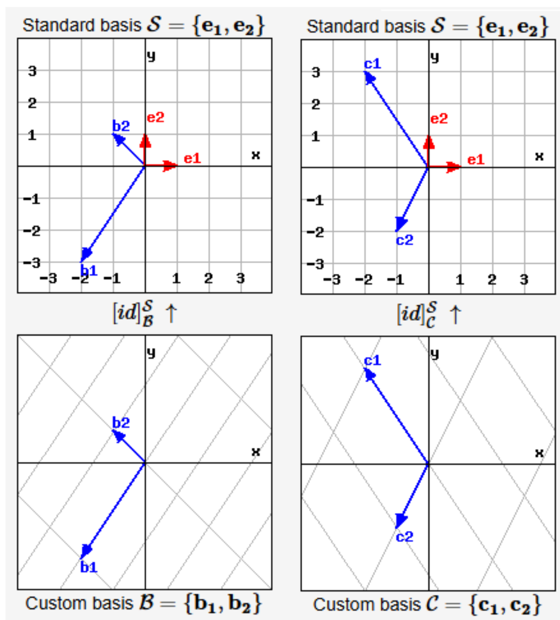
Student ID: ______________________

Duration: 105 Minutes

Grade Table

Question:	1	2	3	4	Total
Points:	10	14	4	8	36
Score:					

1. (a) (7 points) (WebWork) The standard basis $\mathcal{S} = \{e_1, e_2\}$ and two custom bases $\mathcal{B} = \{b_1, b_2\}$ and $\mathcal{C} = \{c_1, c_2\}$ for $\mathbb{R}^2$ are shown in the figures below.



(i) Find the transition matrix from $\mathcal{B}-$coordinates to $\mathcal{C}-$coordinates.

Let $\mathbf{v} \in \mathbb{R}^2$ have coordinates with respect to the basis $\mathcal{B}$ and $\mathcal{C}$

$$[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}, \qquad [\mathbf{v}]_{\mathcal{C}} = \begin{bmatrix} \gamma \\ \delta \end{bmatrix},$$

respectively. Then

$$\mathbf{v} = \alpha \mathbf{b}_1 + \beta \mathbf{b}_2 = \gamma \mathbf{c}_1 + \delta \mathbf{c}_2. \quad (\star)$$

From the figure,

$$\mathbf{b}_1 = \begin{bmatrix} -2 \\ -3 \end{bmatrix}, \qquad \mathbf{b}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \qquad \mathbf{c}_1 = \begin{bmatrix} -2 \\ 3 \end{bmatrix}, \qquad \mathbf{c}_2 = \begin{bmatrix} -1 \\ -2 \end{bmatrix}.$$

Thus, it follows from $(\star)$ that

$$\underbrace{\begin{bmatrix} -2 & -1 \\ -3 & 1 \end{bmatrix}}_{U} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \underbrace{\begin{bmatrix} -2 & -1 \\ 3 & -2 \end{bmatrix}}_{V} \begin{bmatrix} \gamma \\ \delta \end{bmatrix}.$$

Hence, the transition matrix from $\mathcal{B}-$coordinates to $\mathcal{C}-$coordinates is

$$T_{\mathcal{B}\to\mathcal{C}} = V^{-1}U$$

Step 1: We first compute V^{-1} using Gauss–Jordan elimination:

$$\left[\begin{array}{cc|cc} -2 & -1 & 1 & 0 \\ 3 & -2 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \to -\frac{1}{2}R_1} \left[\begin{array}{cc|cc} 1 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 3 & -2 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \to R_2 - 3R_1} \left[\begin{array}{cc|cc} 1 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{7}{2} & \frac{3}{2} & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \to -\frac{2}{7}R_2} \left[\begin{array}{cc|cc} 1 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{3}{7} & -\frac{2}{7} \end{array}\right] \xrightarrow{R_1 \to R_1 - \frac{1}{2}R_2} \left[\begin{array}{cc|cc} 1 & 0 & -\frac{2}{7} & \frac{1}{7} \\ 0 & 1 & -\frac{3}{7} & -\frac{2}{7} \end{array}\right].$$

Therefore,

$$V^{-1} = \begin{bmatrix} -\frac{2}{7} & \frac{1}{7} \\ -\frac{3}{7} & -\frac{2}{7} \end{bmatrix}.$$

Step 2: Now compute

$$T_{\mathcal{B}\to\mathcal{C}} = V^{-1}U = \begin{bmatrix} -\frac{2}{7} & \frac{1}{7} \\ -\frac{3}{7} & -\frac{2}{7} \end{bmatrix} \begin{bmatrix} -2 & -1 \\ -3 & 1 \end{bmatrix}.$$

Carrying out the matrix multiplication,

$$T_{\mathcal{B}\to\mathcal{C}} = \begin{bmatrix} \left(-\frac{2}{7}\right)(-2) + \left(\frac{1}{7}\right)(-3) & \left(-\frac{2}{7}\right)(-1) + \left(\frac{1}{7}\right)(1) \\ \left(-\frac{3}{7}\right)(-2) + \left(-\frac{2}{7}\right)(-3) & \left(-\frac{3}{7}\right)(-1) + \left(-\frac{2}{7}\right)(1) \end{bmatrix} = \begin{bmatrix} \frac{1}{7} & \frac{3}{7} \\ \frac{12}{7} & \frac{1}{7} \end{bmatrix}.$$

Hence, the transition matrix from $\mathcal{B}$-coordinates to $\mathcal{C}$-coordinates is

$$\boxed{T_{\mathcal{B}\to\mathcal{C}} = \begin{bmatrix} \frac{1}{7} & \frac{3}{7} \\ \frac{12}{7} & \frac{1}{7} \end{bmatrix}.}$$

(ii) Find the $\mathcal{B}$-coordinates of the vector $\begin{bmatrix} 5 \\ 6 \end{bmatrix}_{\mathcal{S}}$.

We want to find scalars α and β such that

$$\alpha\, \mathbf{b}_1 + \beta\, \mathbf{b}_2 = 5\mathbf{e}_1 + 6\mathbf{e}_2.$$

This leads to the system

$$\begin{bmatrix} -2 & -1 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

Reducing this augmented matrix to reduced row echelon form, we obtain

$$\left[\begin{array}{cc|c} -2 & -1 & 5 \\ -3 & 1 & 6 \end{array}\right] \xrightarrow{R_2 \to 2R_2 - 3R_1} \left[\begin{array}{cc|c} -2 & -1 & 5 \\ 0 & 5 & -3 \end{array}\right] \xrightarrow{R_2 \to \frac{1}{5}R_2} \left[\begin{array}{cc|c} -2 & -1 & 5 \\ 0 & 1 & -\frac{3}{5} \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 + R_2} \left[\begin{array}{cc|c} -2 & 0 & \frac{22}{5} \\ 0 & 1 & -\frac{3}{5} \end{array}\right] \xrightarrow{R_1 \to -\frac{1}{2}R_1} \left[\begin{array}{cc|c} 1 & 0 & -\frac{11}{5} \\ 0 & 1 & -\frac{3}{5} \end{array}\right].$$

Thus, $\alpha = -\frac{11}{5}, \beta = -\frac{3}{5}$ and

$$\boxed{\begin{bmatrix} 5 \\ 6 \end{bmatrix}_{\mathcal{S}} = \begin{bmatrix} -\frac{11}{5} \\ -\frac{3}{5} \end{bmatrix}_{\mathcal{B}}.}$$

(iii) Find the $\mathcal{S}$-coordinates of the vector $\begin{bmatrix} 2 \\ -3 \end{bmatrix}_{\mathcal{C}}$.

We want to find scalars α and β such that

$$\alpha\,\mathbf{e}_1 + \beta\,\mathbf{e}_2 = 2\mathbf{c}_1 - 3\mathbf{c}_2.$$

This leads to the system

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \end{bmatrix}.$$

Computing the right-hand side, we obtain

$$\begin{bmatrix} -2 & -1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} -4+3 \\ 6+6 \end{bmatrix} = \begin{bmatrix} -1 \\ 12 \end{bmatrix}.$$

Hence,

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} -1 \\ 12 \end{bmatrix}.$$

Therefore,

$$\boxed{\begin{bmatrix} -1 \\ 12 \end{bmatrix}_{\mathcal{S}} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}_{\mathcal{C}}.}$$

(b) (3 points) (WebWork) Let

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \qquad \mathbf{a}_2 = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 5 \\ 5 \\ 0 \end{bmatrix}.$$

Is $\mathbf{b}$ in the span of $\mathbf{a}_1$ and $\mathbf{a}_2$? Provide details.

We want to determine whether there exist scalars c_1 and c_2 such that

$$c_1\mathbf{a}_1 + c_2\mathbf{a}_2 = \mathbf{b}. \quad (\star)$$

This gives

$$c_1 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ 0 \end{bmatrix},$$

so the augmented matrix is

$$\left[\begin{array}{cc|c} 1 & 3 & 5 \\ 2 & 1 & 5 \\ -1 & 4 & 0 \end{array}\right].$$

Now row reduce:

$$\left[\begin{array}{cc|c} 1 & 3 & 5 \\ 2 & 1 & 5 \\ -1 & 4 & 0 \end{array}\right] \xrightarrow[R_3 \to R_3 + R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{cc|c} 1 & 3 & 5 \\ 0 & -5 & -5 \\ 0 & 7 & 5 \end{array}\right] \xrightarrow{R_3 \to 5R_3 + 7R_2} \left[\begin{array}{cc|c} 1 & 3 & 5 \\ 0 & -5 & -5 \\ 0 & 0 & -10 \end{array}\right].$$

The last row yields a contradiction:

$$[0 \;\; 0 \mid c], \qquad c = -10 \neq 0.$$

Therefore, the system is inconsistent, i.e., there are no scalars c_1, c_2 satisfying $(\star)$. Hence

$$\boxed{\mathbf{b} \notin \operatorname{span}\{\mathbf{a}_1, \mathbf{a}_2\}.}$$

2. Answer the following questions in one or two sentences.

(a) (3 points) Let A be a 4×7 matrix with $\text{rank}(A) = 4$ and let $b \in \mathbb{R}^4$. Explain why the system $Ax = b$ must be consistent. Explain why the system must have infinitely many solutions.

Since $\text{rank}(A) = 4$, the matrix A has a pivot in every row. Therefore, for every $b \in \mathbb{R}^4$, the augmented system $Ax = b$ cannot produce a contradictory row, so the system is always consistent.

Also, A has 7 columns and rank 4, so there are $7 - 4 = 3$ free variables. Hence the system has infinitely many solutions.

(b) (2 points) Suppose that a square matrix A has a nonzero vector in its null space. Can you conclude that A is row-equivalent to the identity matrix?

No. If A has a nonzero vector in its null space, then the homogeneous system $Ax = 0$ has a nontrivial solution. This means that the columns of A are linearly dependent, so A cannot have a pivot in every column.

Therefore, A cannot be row-equivalent to the identity matrix.

(c) (2 points) Explain why the transition matrix from one basis to another must be nonsingular.

A transition matrix is formed from the coordinate vectors of basis elements. Since basis vectors are linearly independent, these coordinate vectors are linearly independent. Therefore, the columns of the transition matrix are linearly independent, and hence the matrix is nonsingular.

(d) (2 points) Let E be a product of a sequence of elementary matrices. Is it true that the system $Ex = 0$ has a unique solution?

Yes. Every elementary matrix is nonsingular, and the product of nonsingular matrices is again nonsingular. Therefore, E is nonsingular.

Hence the homogeneous system $Ex = 0$ has the unique solution $x = 0$.

(e) (3 points) Let $\{v_1, v_2, v_3\}$ be a basis for a vector space V, and let $v_4 \in V$. If v_4 is added to this set, is the new set still be a basis for V? Is it still be a spanning set for V?

The new set will still be a spanning set for V, because $\{v_1, v_2, v_3\}$ already spans V, and adding one more vector cannot destroy that property.

However, the new set will not be a basis for V, because $v_4 \in V$ can already be written as a linear combination of v_1, v_2, v_3. Thus the enlarged set is linearly dependent.

(f) (2 points) Does transposing a matrix change its rank?

No. The rank of a matrix is the dimension of its row space, and the rows of A^T are exactly the columns of A. Since the dimensions of the row space and column space of a matrix are equal, A and A^T have the same rank.

3. (4 points) Express the following matrix as a product of elementary matrices:

$$A = \begin{bmatrix} 3 & 2 & -3 \\ 1 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

To express A as a product of elementary matrices, we first row reduce A to the identity matrix:

$$A = \begin{bmatrix} 3 & 2 & -3 \\ 1 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 0 & 0 \\ 3 & 2 & -3 \\ -1 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 - 3R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -3 \\ -1 & 0 & 1 \end{bmatrix} \xrightarrow{R_3 \leftarrow R_3 + R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_2 \leftarrow \frac{1}{2} R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{3}{2} \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 + \frac{3}{2} R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I.$$

Thus,

$$E_5 E_4 E_3 E_2 E_1 A = I,$$

and hence

$$A = E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1} E_5^{-1}.$$

The corresponding inverse elementary matrices are

$$E_1^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix},$$

$$E_4^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad E_5^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{3}{2} \\ 0 & 0 & 1 \end{bmatrix}.$$

Therefore,

$$\boxed{A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{3}{2} \\ 0 & 0 & 1 \end{bmatrix}.}$$

4. (8 points) Consider the following system of linear equations:

$$\begin{cases} x_1 + x_2 + x_3 = 2, \\ 2x_1 + 3x_2 + x_3 = 3, \\ x_1 + 2x_2 = 1, \\ 3x_1 + 5x_2 + x_3 = 4. \end{cases}$$

Use Gauss–Jordan elimination to reduce the augmented matrix of the system to reduced row echelon form. Then determine whether the system is consistent or inconsistent. If it is consistent, determine whether it has a unique solution or infinitely many solutions. If it has infinitely many solutions, determine the dimension of the solution set.

We write the augmented matrix of the system as

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & 3 & 1 & 3 \\ 1 & 2 & 0 & 1 \\ 3 & 5 & 1 & 4 \end{array}\right].$$

Now we perform Gauss–Jordan elimination:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & 3 & 1 & 3 \\ 1 & 2 & 0 & 1 \\ 3 & 5 & 1 & 4 \end{array}\right] \xrightarrow{\substack{R_2 \leftarrow R_2 - 2R_1 \\ R_3 \leftarrow R_3 - R_1 \\ R_4 \leftarrow R_4 - 3R_1}} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 1 & -1 & -1 \\ 0 & 2 & -2 & -2 \end{array}\right]$$

$$\xrightarrow{\substack{R_3 \leftarrow R_3 - R_2 \\ R_4 \leftarrow R_4 - 2R_2}} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 \leftarrow R_1 - R_2} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Thus, the reduced row-echelon form is

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Since there is no row of the form

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & c \end{array}\right] \qquad \text{with } c \neq 0,$$

the system is consistent.

From the reduced row-echelon form, the pivot variables are x_1 and x_2, while x_3 is a free variable. Therefore, the system has infinitely many solutions.

Let

$$x_3 = t, \qquad t \in \mathbb{R}.$$

Then the first two rows give

$$x_1 + 2x_3 = 3 \implies x_1 = 3 - 2t,$$
$$x_2 - x_3 = -1 \implies x_2 = t - 1.$$

Hence the solution set is

$$\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 - 2t \\ t - 1 \\ t \end{bmatrix} : t \in \mathbb{R} \right\}.$$

Since there is one free variable, the dimension of the solution set is 1. Therefore,

$$\boxed{\text{the system is consistent, it has infinitely many solutions, and the solution set has dimension 1.}}$$