23.06.2026

İzmir Institute of Technology
Math 265 Basic Linear Algebra, Spring 2026
Final Exam - Solution Key

Name: ______________________________

Student ID: ______________________________

Duration: 135 Minutes

Grade Table

Question:	1	2	3	Total
Points:	20	12	16	48
Score:				

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1. (WebWork)

(a) (4 points) Consider the following Gaussian elimination:

$$A \longrightarrow \underbrace{\begin{pmatrix} 1 & 0 & 8 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{E_1} A \longrightarrow \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 6 \end{pmatrix}}_{E_2} E_1 A \longrightarrow \underbrace{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}}_{E_3} E_2 E_1 A \longrightarrow \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{E_4} E_3 E_2 E_1 A = \begin{pmatrix} -9 & 5 & 7 \\ 0 & -5 & 6 \\ 0 & 0 & -6 \end{pmatrix}.$$

What is the determinant of A?

Let

$$U = \begin{pmatrix} -9 & 5 & 7 \\ 0 & -5 & 6 \\ 0 & 0 & -6 \end{pmatrix}$$

be the row-reduced matrix obtained from A by the given elementary row operations. Then

$$U = E_4 E_3 E_2 E_1 A.$$

Taking determinants of both sides, we get

$$\det(U) = \det(E_4)\det(E_3)\det(E_2)\det(E_1)\det(A).$$

Since U is upper triangular,

$$\det(U) = (-9)(-5)(-6) = -270.$$

Moreover, since E_1, E_2, E_3, and E_4 are elementary matrices, their determinants are,

$$\det(E_1) = 1, \qquad \det(E_2) = 6, \qquad \det(E_3) = -1, \qquad \det(E_4) = -4.$$

Therefore,

$$-270 = (-4)(-1)(6)(1)\det(A).$$

Hence

$$\det(A) = -\frac{45}{4}.$$

(b) (4 points) Let W be the subspace spanned by the vectors

$$u_1 = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \qquad \text{and} \qquad u_2 = \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix}.$$

Given

$$v = \begin{bmatrix} 4 \\ -13 \\ 7 \end{bmatrix} \in W,$$

find the coordinates of v with respect to the ordered basis c.

First, we observe that

$$u_1 \cdot u_2 = 2 - 5 + 3 = 0,$$

so the vectors u_1 and u_2 are orthogonal. Therefore, $\{u_1, u_2\}$ is an orthogonal basis of W and if

$$v = c_1 u_1 + c_2 u_2,$$

then the coordinates of v with respect to the ordered basis $\{u_1, u_2\}$ are given by

$$c_1 = \frac{v \cdot u_1}{u_1 \cdot u_1}, \qquad c_2 = \frac{v \cdot u_2}{u_2 \cdot u_2}.$$

Computing the inner products for c_1, we get,

$$v \cdot u_1 = 8 + 13 + 21 = 42, \quad u_1 \cdot u_1 = 2^2 + (-1)^2 + 3^2 = 14.$$

Hence

$$c_1 = \frac{42}{14} = 3.$$

Similarly, we compute the inner products for c_2 to get

$$v \cdot u_2 = 4 - 65 + 7 = -54, \quad u_2 \cdot u_2 = 1^2 + 5^2 + 1^2 = 27.$$

Thus

$$c_2 = \frac{-54}{27} = -2.$$

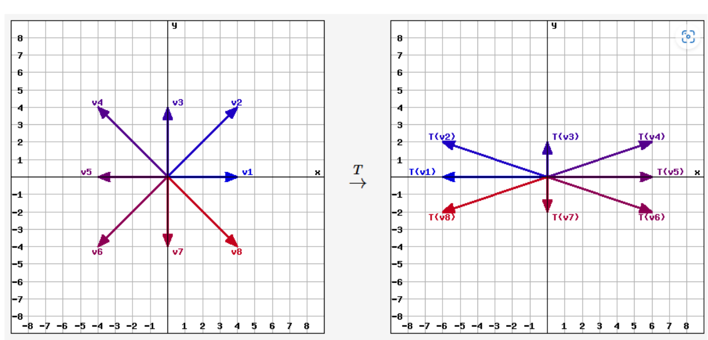
Therefore,

$$v = 3u_1 - 2u_2.$$

So the coordinate vector of v with respect to the ordered basis $\{u_1, u_2\}$ is

$$[v]_{\{u_1,u_2\}} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}.$$

(c) (4 points) Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is the linear transformation defined in the figure below. The figure shows where T maps eight vectors $\mathbf{v}_1, \ldots, \mathbf{v}_8$ from the domain.



Find the eigenvalues of the matrix associated with the transformation T, and find the corresponding eigenspaces.

From the figure, we observe that the horizontal vectors are mapped to horizontal vectors. More precisely,

$$v_1 = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

is mapped to

$$T(v_1) = \begin{bmatrix} -6 \\ 0 \end{bmatrix} = -\frac{3}{2}\begin{bmatrix} 4 \\ 0 \end{bmatrix} = -\frac{3}{2}v_1.$$

Hence $\lambda = -\frac{3}{2}$ is an eigenvalue. Since the eigenvectors lie on the x-axis, the corresponding eigenspace is

$$E_{-\frac{3}{2}} = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}.$$

Similarly, the vertical vectors are mapped to vertical vectors. For example,

$$v_3 = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

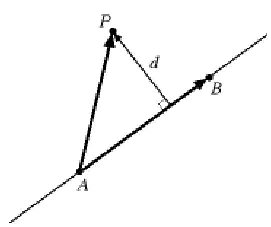
is mapped to

$$T(v_3) = \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 \\ 4 \end{bmatrix} = \frac{1}{2}v_3.$$

Therefore, $\lambda = \frac{1}{2}$ is another eigenvalue and the corresponding eigenspace is

$$E_{\frac{1}{2}} = \operatorname{span}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}.$$

(d) (4 points) The distance d of a point P to the line through points A and B is the length of the component of $\overline{AP}$ that is orthogonal to $\overline{AB}$, as indicated in the figure below.



Find the distance from $P = (-5, -4, 5)$ to the line through the points $A = (0, -2, 4)$ and $B = (-4, 3, -2)$.
Hint: Use the scalar projection.

The distance can be computed by using the scalar projection onto the vector $\overrightarrow{AB}$ and then applying the Pythagorean theorem:

$$d^2 = \|\overrightarrow{AP}\|^2 - \left(\operatorname{comp}_{\overrightarrow{AB}} \overrightarrow{AP}\right)^2. \quad (\star)$$

We have

$$\overrightarrow{AB} = \begin{bmatrix} -4 \\ 5 \\ -6 \end{bmatrix}, \quad \text{and} \quad \overrightarrow{AP} = \begin{bmatrix} -5 \\ -2 \\ 1 \end{bmatrix}.$$

The scalar projection of $\overrightarrow{AP}$ onto $\overrightarrow{AB}$ is

$$\operatorname{comp}_{\overrightarrow{AB}} \overrightarrow{AP} = \frac{\overrightarrow{AP} \cdot \overrightarrow{AB}}{\|\overrightarrow{AB}\|} = \frac{4}{\sqrt{77}}.$$

Moreover,

$$\|\overrightarrow{AP}\|^2 = (-5)^2 + (-2)^2 + 1^2 = 30.$$

It follows from the relation $(\star)$ that

$$d = \sqrt{30^2 - \frac{16}{77}} = \sqrt{\frac{2294}{77}}.$$

(e) (4 points) Find a 2×2 matrix A for which

$$E_{-5} = \operatorname{span}\left\{\begin{bmatrix}2\\1\end{bmatrix}\right\}, \qquad E_3 = \operatorname{span}\left\{\begin{bmatrix}5\\3\end{bmatrix}\right\},$$

where E_λ denotes the eigenspace associated with the eigenvalue λ.

The given eigenspaces provide eigenvectors corresponding to the eigenvalues -5 and 3. Namely, let

$$v_1 = \begin{bmatrix}2\\1\end{bmatrix}, \qquad v_2 = \begin{bmatrix}5\\3\end{bmatrix}.$$

Then v_1 is an eigenvector corresponding to the eigenvalue -5, and v_2 is an eigenvector corresponding to the eigenvalue 3. Moreover,

$$\det\begin{pmatrix}2 & 5\\1 & 3\end{pmatrix} = 6 - 5 = 1 \neq 0. \quad (\star)$$

Therefore, these two eigenvectors are linearly independent. Thus, we can use the diagonalization argument and write

$$A = PDP^{-1},$$

where

$$P = \begin{pmatrix}| & |\\v_1 & v_2\\| & |\end{pmatrix} = \begin{pmatrix}2 & 5\\1 & 3\end{pmatrix}, \qquad D = \begin{pmatrix}\lambda_1 & 0\\0 & \lambda_2\end{pmatrix} = \begin{pmatrix}-5 & 0\\0 & 3\end{pmatrix}.$$

Moreover, thanks to $(\star)$, P is nonsingular and given by

$$P^{-1} = \frac{1}{\det P}\begin{pmatrix}3 & -5\\-1 & 2\end{pmatrix} = \begin{pmatrix}3 & -5\\-1 & 2\end{pmatrix}.$$

Therefore,

$$A = PDP^{-1} = \begin{pmatrix}2 & 5\\1 & 3\end{pmatrix}\begin{pmatrix}-5 & 0\\0 & 3\end{pmatrix}\begin{pmatrix}3 & -5\\-1 & 2\end{pmatrix}.$$

Performing the multiplication, we find

$$A = \begin{pmatrix}-45 & 80\\-24 & 43\end{pmatrix}.$$

2. Answer the following questions in two or three sentences. Explain your reasoning clearly.

(a) (3 points) Let $\{u_1, u_2\}$ be an orthonormal basis of $\mathbb{R}^2$ and $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a rotation. Is it true that $\{Tu_1, Tu_2\}$ is also an orthonormal basis of $\mathbb{R}^2$?

Yes. A rotation preserves lengths and angles. Since $\{u_1, u_2\}$ is an orthonormal set, $\{Tu_1, Tu_2\}$ is also an orthonormal set, therefore a basis of $\mathbb{R}^2$.

(b) (3 points) Suppose that during the Gram-Schmidt process, one of the obtained vectors becomes the zero vector. What does this tell us about the original set of vectors?

Let $u_1, u_2, \ldots, u_n$ be the original vectors, and let $v_1, v_2, \ldots, v_n$ be the vectors obtained by the Gram-Schmidt process. Suppose that, for some k, we obtain

$$v_k = 0.$$

In the Gram-Schmidt process, v_k is obtained by subtracting from u_k its projections onto the previous orthogonal vectors:

$$v_k = u_k - \mathrm{proj}_{v_1}(u_k) - \mathrm{proj}_{v_2}(u_k) - \cdots - \mathrm{proj}_{v_{k-1}}(u_k).$$

If $v_k = 0$, then

$$u_k = \mathrm{proj}_{v_1}(u_k) + \mathrm{proj}_{v_2}(u_k) + \cdots + \mathrm{proj}_{v_{k-1}}(u_k).$$

That is, u_k is a linear combination of $v_1, \ldots, v_{k-1}$. However,

$$\mathrm{span}\{v_1, \ldots, v_{k-1}\} = \mathrm{span}\{u_1, \ldots, u_{k-1}\}.$$

Thus, u_k is also a linear combination of the previous vectors $u_1, \ldots, u_{k-1}$.
Therefore, if $v_k = 0$, then the original set of vectors is linearly dependent.

(c) (3 points) Suppose that a 2×2 matrix A has only one eigenvalue λ with algebraic multiplicity 2. Does this imply that A is diagonalizable?

No. If a 2×2 matrix has only one eigenvalue λ with algebraic multiplicity 2, then the dimension of the corresponding eigenspace may be either 1 or 2. Therefore, the statement is not true in general.

(d) (3 points) Is it true that if 0 is an eigenvalue of a square matrix A, then A is singular? What about the converse? That is, if A is singular, is it true that 0 is an eigenvalue?

Yes. If 0 is an eigenvalue of A, then there exists a nonzero vector v such that $Av = 0v = 0$. Thus, $Av = 0$ has a nontrivial solution, which implies A is singular. The converse is also true and can be shown by following similar arguments.

3. Let $\mathcal{V}$ denote the vector space spanned by the functions

$$\{1, \cos x, \cos 2x, \cos 3x\},$$

and let

$$\mathcal{S} = \{1, \cos x, \cos 2x, \cos 3x\}$$

be its ordered basis. We represent

$$f(x) = a_0 + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x \in \mathcal{V}$$

by the coordinate vector

$$[f]_{\mathcal{S}} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} \in \mathbb{R}^4.$$

For example, if

$$f(x) = 2 - 3\cos x + 4\cos 3x,$$

then

$$[f]_{\mathcal{S}} = \begin{bmatrix} 2 \\ -3 \\ 0 \\ 4 \end{bmatrix}.$$

Using the vector representations of functions described above, answer the following questions.

(a) (4 points) Write the vector representation of each basis element in $\mathcal{S}$.

The ordered basis is

$$\mathcal{S} = \{1, \cos x, \cos 2x, \cos 3x\}.$$

Using the given coordinate convention, we have

$$[1]_{\mathcal{S}} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad [\cos x]_{\mathcal{S}} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \qquad [\cos 2x]_{\mathcal{S}} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \qquad [\cos 3x]_{\mathcal{S}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

(b) (4 points) Consider the second derivative operator

$$\frac{d^2}{dx^2} : \mathcal{V} \to \mathcal{V}, \qquad f \mapsto f''.$$

Let D be the matrix representation of this operator with respect to the vector representations described above. Find D.

Let

$$f(x) = a_0 + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x.$$

Then, vector representation of f in $\mathcal{S}$ is

$$[f]_{\mathcal{S}} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}.$$

The second-order derivative is

$$f''(x) = -a_1 \cos x - 4a_2 \cos 2x - 9a_3 \cos 3x.$$

With respect to the ordered basis $\{1, \cos x, \cos 2x, \cos 3x\}$ of $\mathcal{V}$, this is represented by

$$[f'']_{\mathcal{S}} = \begin{bmatrix} 0 \\ -a_1 \\ -4a_2 \\ -9a_3 \end{bmatrix}.$$

Thus, we want to find a matrix D such that

$$D\left(\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -a_1 \\ -4a_2 \\ -9a_3 \end{bmatrix}.$$

To construct D, we look at the images of the basis vectors we find in part (a).

$$De_1 = D\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad De_2 = D\left(\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \quad De_3 = D\left(\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ -4 \\ 0 \end{bmatrix}, \quad De_4 = D\left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -9 \end{bmatrix}.$$

Therefore, the columns of D are De_1, De_2, De_3, De_4. Hence

$$D = \begin{bmatrix} | & | & | & | \\ De_1 & De_2 & De_3 & De_4 \\ | & | & | & | \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & -9 \end{bmatrix}.$$

(c) (2 points) Find $\ker(D)$ and discuss whether the operator D is injective.
Recall that D represents the second derivative operator on $\mathcal{V}$. Therefore,

$$f \in \ker(D) \iff f'' = 0.$$

Among the functions in

$$\mathcal{V} = \operatorname{span}\{1, \cos x, \cos 2x, \cos 3x\},$$

the only functions whose second derivative is zero are the constant functions. Hence, in coordinate form,

$$\ker(D) = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right\}.$$

Since $\ker(D) \neq \{\mathbf{0}\}$, the operator D is not injective.

(d) (2 points) Find the dimension of the range of D and discuss whether the operator D is surjective.
Hint: Use your answer in part (c) and the rank-nullity theorem.
From part (c), we find

$$\ker(D) = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right\}.$$

Hence

$$\dim(\ker(D)) = 1.$$

Since $\dim(\mathcal{V}) = 4$, by the rank-nullity theorem,

$$\dim(\mathcal{V}) = \dim(\ker(D)) + \dim(\text{Range}(D)).$$

Therefore,

$$4 = 1 + \dim(\text{Range}(D)) \Rightarrow \dim(\text{Range}(D)) = 3.$$

Thus, the operator D is not surjective.

(e) (4 points) What are the eigenvalues of D? Which basis elements are eigenvectors of D?
Since D is diagonal, its eigenvalues are the diagonal entries. Hence the eigenvalues of D are

$$\lambda_1 = 0, \quad \lambda_2 = -1, \quad \lambda_3 = -4, \quad \lambda_4 = -9.$$

The vector representations of the basis elements are

$$[1]_{\mathcal{S}} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad [\cos x]_{\mathcal{S}} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \qquad [\cos 2x]_{\mathcal{S}} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \qquad [\cos 3x]_{\mathcal{S}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Moreover, from part (b)

$$D[1]_{\mathcal{S}} = 0[1]_{\mathcal{S}} = \lambda_1[1]_{\mathcal{S}},$$
$$D[\cos x]_{\mathcal{S}} = -[\cos x]_{\mathcal{S}} = \lambda_2[\cos x]_{\mathcal{S}},$$
$$D[\cos 2x]_{\mathcal{S}} = -4[\cos 2x]_{\mathcal{S}} = \lambda_3[\cos 2x]_{\mathcal{S}},$$

and

$$D[\cos 3x]_{\mathcal{S}} = -9[\cos 3x]_{\mathcal{S}} = \lambda_4[\cos 3x]_{\mathcal{S}}.$$

Therefore, all basis elements in $\mathcal{S}$, through their vector representations, are eigenvectors of D.