10.06.2025

İzmir Institute of Technology
MSE 222 Applied Mathematics for Materials Science and Engineering, Spring 2025
Final Examination – Solution Key

Name: ______________________________

Student ID: ______________________________

Duration: 150 Minutes

Grade Table

Question:	1	2	3	4	5	Total
Points:	30	25	25	15	25	120
Score:						

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1. Answer the following questions. Write an explanation (with at most 3 sentences), if required.

(a) (4 points) What is the maximum number of zeros that a 3×3 matrix can have without having a zero determinant? Explain.

Placing seven zeros in into a 3×3 matrix yields a guaranteed zero row or column.

(b) (4 points) If multiplication by A rotates a vector $\mathbf{x}$ in $xy-$plane through an angle θ, what is the effect of multiplying $\mathbf{x}$ by A^T? Explain.

Using $\cos\theta = \cos(-\theta)$ and $\sin\theta = -\sin(-\theta)$

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix}.$$

Hence, A^T rotates a vector in $xy-$plane through an angle θ clockwise.

(c) (4 points) Let E be an $m \times m$ elementary matrix and A be an $m \times n$ matrix. Then, is it true that null space of EA is the same as the null space of A? Explain.

Elementary row operations do not change solution set of a liner system. That is, the linear systems $A\mathbf{x} = 0$ and $EA\mathbf{x} = 0$ assume the same solution set. Therefore, $\text{Null}(A) = \text{Null}(EA)$.

(d) (4 points) Is it true that adding one additional column to a matrix increases its rank? Explain.

Depends whether the additional column lies in the column space or not. If it lies in the column space, then rank of the matrix remains same, otherwise rank of the matrix increases.

(e) (4 points) Let f be a periodic function. Is it true that its derivative f' is also periodic? Explain why or why not.

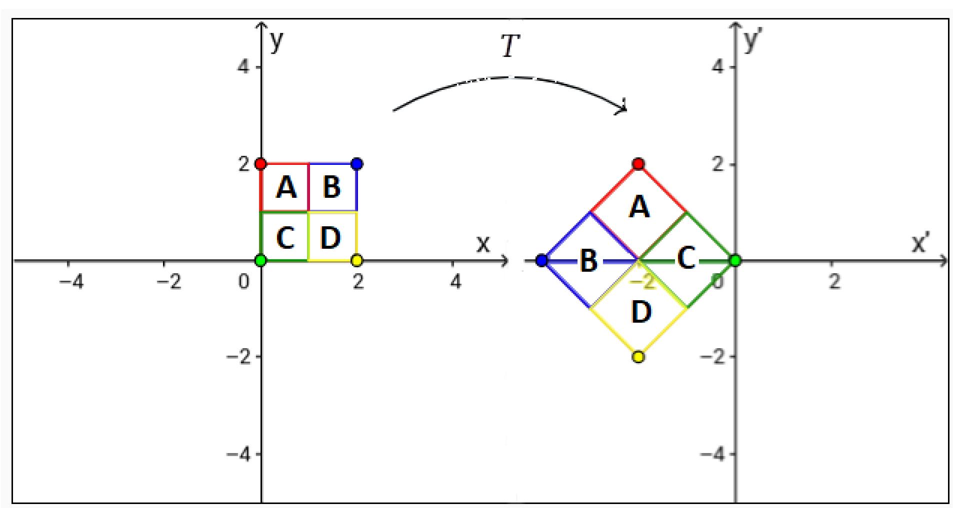
Let f be a $p-$periodic function, that is, $f(x) = f(x+p)$ for all $x \in \mathbb{R}$. Then, $f'(x) = f'(x+p)$.

(f) (5 points) Let f be an even function. Suppose that its first order derivative exists. Is it true that f and its derivative f' are orthogonal with respect to the usual inner product defined on a symmetric interval? Show your work.

$$\langle f, f' \rangle = \int_{-L}^{L} f(x)f'(x)dx \overset{u=f(x)}{=} \int_{x=-L}^{x=L} u du = \frac{u^2}{2}\Big|_{x=-L}^{x=L} = f^2(x) - f^2(-x) = 0.$$

Hence, $f \perp f'$.

(g) (5 points) Find the linear transformation T that maps the region on the left to the region on the right. Show your work.



To find the region on the right, we can do the following operations in that order

- reflection with respect to the $y-$axis: $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$,
- counterclockwise rotation by $\theta = \frac{\pi}{4}$ radians: $\begin{bmatrix} \cos\left(\frac{\pi}{4}\right) & -\sin\left(\frac{\pi}{4}\right) \\ \sin\left(\frac{\pi}{4}\right) & \cos\left(\frac{\pi}{4}\right) \end{bmatrix}$,
- scaling by $\sqrt{2}$ times: $\begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix}$.

Hence,

$$T = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} \cos\left(\frac{\pi}{4}\right) & -\sin\left(\frac{\pi}{4}\right) \\ \sin\left(\frac{\pi}{4}\right) & \cos\left(\frac{\pi}{4}\right) \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}.$$

2. Consider the following 2−periodic function

$$f(x) = \begin{cases} -1, & -1 \leq x < 0, \\ 1, & 0 \leq x < 1; \end{cases} \qquad f(x) = f(x+2)$$

(a) (20 points) Find the Fourier series expansion of f.

(b) (5 points) What is the value of the Fourier series expansion evaluated at the point $x = 0$?

(a) f is an odd function. Therefore, the constant term and the coefficients of cosine components of the Fourier series expansion is 0:

$$a_0 = 0, \quad a_n = 0, n \in \mathbb{Z}^+.$$

Coefficients of the sine components are evaluated as

$$\begin{aligned} b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx \\ &= \int_{-1}^{1} f(x)\sin(n\pi x)dx \\ &= 2\int_{0}^{1} f(x)\sin(n\pi x)dx \\ &= 2\int_{0}^{1} \sin(n\pi x)dx \\ &= 2\left(-\frac{\cos(n\pi x)}{n\pi}\right)\Bigg|_{x=0}^{x=1} \\ &= \frac{2(1-(-1)^n)}{n\pi}, \quad n \in \mathbb{Z}^+ \end{aligned}$$

Observe that for even indexed terms, $b_n = 0$. Therefore, changing the index as $n = 2k-1$, $k \in \mathbb{Z}^+$, we can write the coefficients b_k in the index k as

$$b_k = \frac{2(1-(-1)^{2k-1})}{(2k-1)\pi} = \frac{4}{(2k-1)\pi}$$

Hence, the Fourier series expansion of f is given by

$$\sum_{n=1}^{\infty} b_n \sin(n\pi x) = \sum_{k=1}^{\infty} b_k \sin((2k-1)\pi x) = \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{2k-1}\sin((2k-1)\pi x).$$

(b) f is discontinuous at $x = 0$. Therefore, at $x = 0$, the Fourier series converges to

$$\frac{\lim_{x\to 0^-} f(x) + \lim_{x\to 0^+} f(x)}{2} = \frac{-1+1}{2} = 0.$$

3. (25 points) Consider the space $C([-\pi,\pi])$ with the usual inner product defined by

$$\langle f,g\rangle = \int_{-\pi}^{\pi} f(x)g(x)dx, \quad \text{for all } f,g \in C([-\pi,\pi]).$$

Use the Gram-Schmidt process to create an orthogonal basis for the subspace of $C([-\pi,\pi])$ spanned by the set of functions $\{\sin x, \sin^2 x, \sin^3 x\}$.

Let ϕ_1, ϕ_2 and ϕ_3 be the orthogonal functions that the Gram-Schmidt process defines. Then,

$$\phi_1(x) = \sin x.$$

Next, we get

$$\begin{aligned}\phi_2(x) &= \sin^2 x - \frac{\langle \sin^2 x, \varphi_1(x)\rangle}{\langle \varphi_1(x), \varphi_1(x)\rangle}\varphi_1(x) \\ &= \sin^2 x - \frac{\cancelto{0}{\int_{-\pi}^{\pi} \sin^2 x \sin x dx}}{\int_{-\pi}^{\pi} \sin x \sin x dx} = \sin^2 x.\end{aligned}$$

Finally, we write ϕ_3 as

$$\begin{aligned}\phi_3(x) &= \sin^3 x - \frac{\langle \sin^3 x, \varphi_1(x)\rangle}{\langle \varphi_1(x), \varphi_1(x)\rangle}\varphi_1(x) - \frac{\langle \sin^3 x, \varphi_2(x)\rangle}{\langle \varphi_2(x), \varphi_2(x)\rangle}\varphi_2(x) \\ &= \sin^3 x - \frac{\int_{-\pi}^{\pi} \sin^3 x \sin x dx}{\int_{-\pi}^{\pi} \sin x \sin x dx}\sin x - \frac{\cancelto{0}{\int_{-\pi}^{\pi} \sin^3 x \sin^2 x dx}}{\int_{-\pi}^{\pi} \sin^2 x \sin^2 x dx}\sin^2 x \\ &= \sin^3 x - \frac{\int_{-\pi}^{\pi} \sin^4 x dx}{\int_{-\pi}^{\pi} \sin^2 x dx}\sin x.\end{aligned}$$

Using half-angle formula, integrals in the above result can be evaluated as follows:

$$\int_{-\pi}^{\pi} \sin^2 x dx = \int_{-\pi}^{\pi} \frac{1-\cos(2x)}{2}dx = \left(\frac{x}{2} - \frac{\sin(2x)}{4}\right)\Bigg|_{x=-\pi}^{x=\pi} = \pi$$

and

$$\begin{aligned}\int_{-\pi}^{\pi} \sin^4 x dx &= \int_{-\pi}^{\pi} \left(\frac{1-\cos(2x)}{2}\right)^2 dx \\ &= \int_{-\pi}^{\pi} \frac{1}{4}dx - \int_{-\pi}^{\pi} \cos(2x)dx + \int_{-\pi}^{\pi} \frac{\cos^2(2x)}{4}dx \\ &= \frac{\pi}{2} - 0 + \int_{-\pi}^{\pi} \frac{1+\cos(4x)}{8}dx \\ &= \frac{\pi}{2} + \left(\frac{x}{8} + \frac{\sin(4x)}{32}\right)\Bigg|_{x=-\pi}^{x=\pi} = \frac{3\pi}{4}.\end{aligned}$$

Hence,

$$\phi_3(x) = \sin^3 x - \frac{3}{4}\sin x.$$

4. (15 points) Show that

$$\left\{\cos\left(\frac{(2n-1)\pi}{2}x\right)\right\}_{n=1}^{\infty}$$

forms an orthogonal set with respect to the usual inner product

$$\langle f,g\rangle = \int_0^1 f(x)g(x)dx, \quad \forall f,g\in C([0,1]).$$

For all $m,n\in\mathbb{Z}^+$, with $m\neq n$, we need to show

$$\int_0^1 \left[\cos\left(\frac{(2n-1)\pi}{2}x\right)\cos\left(\frac{(2m-1)\pi}{2}x\right)\right]dx = 0.$$

Using the trigonometric identity $\cos(mx)\cos(nx) = \frac{1}{2}\left[\cos((m-n)x)+\cos((m+n)x)\right]$ and then direct integration yields

$$\begin{aligned}\int_0^1 \left[\cos\left(\frac{(2n-1)\pi}{2}x\right)\cos\left(\frac{(2m-1)\pi}{2}x\right)\right]dx &= \frac{1}{2}\int_0^1 \cos((n-m)\pi x)dx + \frac{1}{2}\int_0^1 \cos((n+m-1)\pi x)dx \\ &= \frac{1}{2}\frac{\sin((n-m)\pi x)}{(n-m)\pi}\bigg|_{x=0}^{x=1} + \frac{1}{2}\frac{\sin((n+m-1)\pi x)}{(n+m-1)\pi}\bigg|_{x=0}^{x=1} \\ &= 0.\end{aligned}$$

In addition, for $n=m$, we obtain

$$\begin{aligned}\int_0^1 \cos^2\left(\frac{(2n-1)\pi}{2}x\right)dx &= \int_0^1 \frac{1+\cos((2n-1)\pi x)}{2}dx \\ &= \left(\frac{x}{2}+\frac{\sin((2n-1)\pi x)}{2\pi(2n-1)}\right)\bigg|_{x=0}^{x=1} \\ &= \frac{1}{2}.\end{aligned}$$

Hence,

$$\left\langle \cos\left(\frac{(2n-1)\pi}{2}x\right), \cos\left(\frac{(2m-1)\pi}{2}x\right)\right\rangle = \begin{cases} 0, & m\neq 0, \\ \frac{1}{2}, & m=n. \end{cases}$$

5. (25 points) Find the solution of the heat conduction problem

$$\begin{cases} u_t - 4u_{xx} = 0, & 0 < x < 1, t > 0, \\ u_x(0,t) = u(1,t) = 0, & t > 0, \\ u(x,0) = 1 - x, & 0 < x < 1. \end{cases}$$

Assuming a solution of the form $u(x,t) = X(x)T(t)$ yields two ODEs: A boundary-value problem in X

$$\begin{cases} X'' - \lambda X = 0, \\ X'(0) = X(1) = 0, \end{cases}$$

and a first-order ODE in T

$$T' - 4\lambda T = 0.$$

Case $\lambda \geq 0$ yields a trivial solution $X = 0$. So, let us consider the case $\lambda < 0$. Then, general solution to the $X-$problem is

$$X(x) = A\cos\left(\sqrt{-\lambda}x\right) + B\sin\left(\sqrt{-\lambda}x\right)$$

with derivative

$$X'(x) = -A\sqrt{-\lambda}\sin\left(\sqrt{-\lambda}x\right) + B\sqrt{-\lambda}\cos\left(\sqrt{-\lambda}x\right).$$

Employing the first boundary condition, we get

$$X'(0) = 0 \Rightarrow B = 0.$$

Therefore, solution is of the form $X(x) = A\cos\left(\sqrt{-\lambda}x\right)$. Now, the second boundary condition yields

$$X(1) = A\cos\left(\sqrt{-\lambda}\right) = 0 \Rightarrow \sqrt{-\lambda} = \frac{(2n-1)\pi}{2} \Rightarrow \lambda_n = -\frac{(2n-1)^2\pi^2}{4}$$

where n is a positive integer.
Next, we solve the $T-$problem as

$$T(t) = c_n e^{4\lambda_n t} = c_n e^{-(2n-1)^2\pi^2 t},$$

where c_n varies with respect to different values of λ_n which also varies with respect to $n \in \mathbb{Z}^+$. Hence, a single solution is of the form

$$X_n(x)T_n(t) = c_n\cos\left(\frac{(2n-1)\pi}{2}x\right)e^{-(2n-1)^2\pi^2 t}.$$

Now, employing the principle of superposition, we obtain the general solution as

$$u(x,t) = \sum_{n=1}^{\infty} X_n(x)T_n(t) = \sum_{n=1}^{\infty} c_n\cos\left(\frac{(2n-1)\pi}{2}x\right)e^{-(2n-1)^2\pi^2 t}.$$

To find c_n's, first note that for $t = 0$, we have

$$u(x,0) = 1 - x = \sum_{n=1}^{\infty} c_n\cos\left(\frac{(2n-1)\pi}{2}x\right).$$

Next, note that from the previous question, we know that the functions $\cos\left(\frac{(2n-1)\pi}{2}x\right)$, $n \in \mathbb{Z}^+$ form an orthogonal system, that is,

$$\left\langle \cos\left(\frac{(2n-1)\pi}{2}x\right), \cos\left(\frac{(2m-1)\pi}{2}x\right)\right\rangle = \begin{cases} 0, & m \neq 0, \\ \frac{1}{2}, & m = n. \end{cases}$$

Therefore, we take inner product of both sides by $\cos\left(\frac{(2m-1)\pi}{2}x\right)$ to get

$$\left\langle 1 - x, \cos\left(\frac{(2m-1)\pi}{2}x\right)\right\rangle = \left\langle \sum_{n=1}^{\infty} c_n\cos\left(\frac{(2n-1)\pi}{2}x\right), \cos\left(\frac{(2m-1)\pi}{2}x\right)\right\rangle$$

which is

$$\int_0^1 (1-x)\cos\left(\frac{(2m-1)\pi}{2}x\right)dx = \frac{c_n}{2}.$$

Let us evaluate the integral on the left hand side. We set

$$u = 1 - x \quad \Rightarrow \quad du = -dx,$$
$$dv = \cos\left(\frac{(2m-1)\pi}{2}x\right)dx \quad \Rightarrow \quad v = \frac{2}{(2m-1)\pi}\sin\left(\frac{(2m-1)\pi}{2}x\right).$$

Then, we apply integrtion by parts to get

$$\int_0^1 (1-x)\cos\left(\frac{(2m-1)\pi}{2}x\right)dx = \underbrace{(1-x)\frac{2}{(2m-1)\pi}\sin\left(\frac{(2m-1)\pi}{2}x\right)\Bigg|_{x=0}^{x=1}}_{0} - \int_0^1 \frac{2}{(2m-1)\pi}\sin\left(\frac{(2m-1)\pi}{2}x\right)(-1)dx$$
$$= -\frac{4}{(2m-1)^2\pi^2}\cos\left(\frac{(2m-1)\pi}{2}x\right)\Bigg|_{x=0}^{x=1}$$
$$= \frac{4}{(2m-1)^2\pi^2}$$

Hence,

$$c_n = \frac{8}{(2m-1)^2\pi^2}$$

and solution to the initial-boundary value problem is

$$u(x,t) = \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2m-1)^2}\cos\left(\frac{(2n-1)\pi}{2}x\right)e^{-(2n-1)^2\pi^2 t}.$$